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Constant-Cutoff Approach to Soliton Polarizabilities

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The static electromagnetic polarizabilities of nucleons and Δ-particles are calculated in the simplified Skyrme model, where the quartic Skyrme stabilizing term is omitted and where the constant-cutoff stabilization method is used to stabilize the Skyrme soliton. The numerical results are of the same accuracy as those obtained using the complete Skyrme model, but the simplified Skyrme model offers simpler mathematical structure and easier calculations.

1. INTRODUCTION

It was shown by Skyrme (1961, 1962) that baryons can be treated as solitons of a nonlinear chiral theory. The original Lagrangian of the chiral $SU(2)$ σ-model is

$$\mathscr{L} = \frac{F_\pi^2}{16} \operatorname{Tr} \partial_\mu U \, \partial^\mu U^+ \tag{1.1}$$

where

$$U = \frac{2}{F_\pi} (\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \tag{1.2}$$

is a unitary operator ($UU^+ = 1$) and F_π is the pion-decay constant. In (1.2), $\sigma = \sigma(\mathbf{r})$ is a scalar meson field and $\boldsymbol{\pi} = \boldsymbol{\pi}(\mathbf{r})$ is the pion isotriplet.

The classical stability of the soliton solution to the chiral σ-model Lagrangian requires the additional ad hoc term, proposed by Skyrme (1961, 1962) to be added to (1.1),

$$\mathscr{L}_{\text{Sk}} = \frac{1}{32e^2} \operatorname{Tr}[U^+ \partial_\mu U, U^+ \partial_\nu U]^2 \tag{1.3}$$

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with a dimensionless parameter e and where $[A, B] = AB - BA$. It was shown by several authors (Atkins *et al.*, 1983; see also Holzwarth and Schwesinger, 1986, and Nyman and Riska, 1990, and references therein) that, after the collective quantization using the spherically symmetric ansatz

$$U_0(\mathbf{r}) = \exp[i\boldsymbol{\tau} \cdot \mathbf{r}_0 F(r)], \qquad \mathbf{r}_0 = \mathbf{r}/r \tag{1.4}$$

the chiral model, with both (1.1) and (1.3) included, gives good agreement with experiment for several important physical quantities. Thus it should be possible to derive the effective chiral Lagrangian, obtained as a sum of (1.1) and (1.3), from a more fundamental theory like QCD. On the other hand, it is not easy to generate a term like (1.3) and give a clear physical meaning to the dimensionless constant e in (1.3) using QCD.

Mignaco and Wulck (MW) (1989) indicated therefore a possibility to build a stable single-baryon ($n = 1$) quantum state in the simple chiral theory, with the Skyrme stabilizing term (1.3) omitted. MW showed that the chiral angle $F(r)$ is in fact a function of a dimensionless variable $s = \frac{1}{2}\chi''(0)r$, where $\chi''(0)$ is an arbitrary dimensional parameter intimately connected to the usual stability argument against the soliton solution for the nonlinear σ-model Lagrangian.

Using the adiabatically rotated ansatz $U(\mathbf{r}, t) = A(t)U_0(\mathbf{r})A^+(t)$, where $U_0(\mathbf{r})$ is given by (1.4), MW obtained the total energy of the nonlinear σ-model soliton in the form

$$E = \frac{\pi}{4} F_\pi^2 \frac{1}{\chi''(0)} a + \frac{1}{2} \frac{[\chi''(0)]^3}{(\pi/4)F_\pi^2 b} J(J+1) \tag{1.5}$$

where

$$a = \int_0^\infty \left[\frac{1}{4} s^2 \left(\frac{d\mathscr{F}}{ds}\right)^2 + 8 \sin^2\left(\frac{1}{4}\mathscr{F}\right)\right] dr \tag{1.6}$$

$$b = \int_0^\infty ds \frac{64}{3} s^2 \sin^2\left(\frac{1}{4}\mathscr{F}\right) \tag{1.7}$$

and $\mathscr{F}(s)$ is defined by

$$F(r) = F(s) = -n\pi + \frac{1}{4}\mathscr{F}(s) \tag{1.8}$$

The stable minimum of the function (1.5) with respect to the arbitrary dimensional scale parameter $\chi''(0)$ is

$$E = \frac{4}{3} F_\pi \left[\frac{3}{2}\left(\frac{\pi}{4}\right)^2 \frac{a^3}{b} J(J+1)\right]^{1/4} \tag{1.9}$$

Despite the nonexistence of the stable classical soliton solution to the nonlinear σ-model, it is possible, after the collective coordinate quantization, to build a stable chiral soliton at the quantum level, provided that there is a solution $F = F(r)$ which satisfies the soliton boundary conditions, i.e., $F(0) = -n\pi$, $F(\infty) = 0$, such that the integrals (1.6) and (1.7) exist.

However, as pointed out by Iwasaki and Ohyama (1989), the quantum stabilization method in the form proposed by Mignaco and Wulck (1989) is not correct, since in the simple σ-model the conditions $F(0) = -n\pi$ and $F(\infty) = 0$ cannot be satisfied simultaneously. In other words, if the condition $F(0) = -\pi$ is satisfied, Iwasaki and Ohyama obtained numerically $F(\infty) \to -\pi/2$, and the chiral phase $F = F(r)$ with correct boundary conditions does not exist.

Iwasaki and Ohyama also proved analytically that both boundary conditions $F(0) = -n\pi$ and $F(\infty) = 0$ cannot be satisfied simultaneously. Introducing a new variable $y = 1/r$ into the differential equation for the chiral angle $F = F(r)$, we obtain

$$\frac{d^2F}{dy^2} = \frac{1}{y^2} \sin 2F \tag{1.10}$$

There are two kinds of asymptotic solutions to equation (1.10) around the point $y = 0$, which is called a regular singular point if $\sin 2F \approx 2F$. These solutions are

$$F(y) = \frac{m\pi}{2} + cy^2, \qquad m = \text{even integer} \tag{1.11}$$

$$F(y) = \frac{m\pi}{2} + (cy)^{1/2} \cos\left[\frac{\sqrt{7}}{2} \ln(cy) + \alpha\right], \qquad m = \text{odd integer} \tag{1.12}$$

where c is an arbitrary constant and α is a constant to be chosen appropriately. When $F(0) = -n\pi$ then we want to know which of these two solutions is approached by $F(y)$ when $y \to 0$ ($r \to \infty$). In order to answer that question we multiply (1.10) by $y^3F'(y)$, integrate with respect to y from y to ∞, and use $F(0) = -n\pi$. Thus we get

$$y^2F'(y) + \int_y^\infty 2y[F'(y)]^2\, dy = 1 - \cos[2F(y)] \tag{1.13}$$

Since the left-hand side of (1.13) is always positive, the value of $F(y)$ is always limited to the interval $n\pi - \pi < F(y) < n\pi + \pi$. Taking the limit $y \to 0$, we find that (1.13) is reduced to

$$\int_0^\infty 2y[F'(y)]^2\, dy = 1 - (-1)^m \tag{1.14}$$

where we used (1.11)–(1.12). Since the left-hand side of (1.14) is strictly positive, we must choose an odd integer m. Thus the solution satisfying $F(0) = -n\pi$ approaches (1.12) and we have $F(\infty) \neq 0$. The behavior of the solution (1.11) in the asymptotic region $y \to \infty$ $(r \to 0)$ is investigated by multiplying (1.10) by $F'(y)$, integrating from 0 to y, and using (1.11). The result is

$$[F'(y)]^2 = \frac{2\sin^2 F(y)}{y^2} + \int_0^y \frac{2\sin^2 F(y)}{y^3}\,dy \tag{1.15}$$

From (1.15) we see that $F'(y) \to \text{const}$ as $y \to \infty$, which means that $F(r) \simeq 1/r$ for $r \to 0$. This solution has a singularity at the origin and cannot satisfy the usual boundary condition $F(0) = -n\pi$.

In Dalarsson (1991*b*), I suggested a method to resolve this difficulty by introducing a radial modification phase $\varphi = \varphi(r)$ in the ansatz (1.4) as follows:

$$U(\mathbf{r}) = \exp[i\boldsymbol{\tau} \cdot \mathbf{r}_0 F(r) + i\varphi(r)], \qquad \mathbf{r}_0 = \mathbf{r}/r \tag{1.16}$$

Such a method provides a stable chiral quantum soliton, but the resulting model is an entirely noncovariant chiral model, different from the original chiral σ-model.

In the present paper we use the constant-cutoff limit of the cutoff quantization method developed by Balakrishna *et al.* (1991; see also Jain *et al.*, 1989) to construct a stable chiral quantum soliton within the original chiral σ-model. Then we apply this method to the CHK model of strange baryons and derive the results for spectra of hyperons and strange dibaryons.

The reason the cutoff approach to the problem of the chiral quantum soliton works is connected to the fact that the solution $F = F(r)$ which satisfies the boundary condition $F(\infty) = 0$ is singular at $r = 0$. From the physical point of view the chiral quantum model is not applicable to the region about the origin, since in that region there is a quark-dominated bag of the soliton.

However, as argued in Balakrishna *et al.* (1991), when a cutoff ε is introduced then the boundary conditions $F(\varepsilon) = -n\pi$ and $F(\infty) = 0$, can be satisfied. In Balakrishna *et al.* (1991) an interesting analogy with the damped pendulum is discussed, showing clearly that as long as $\varepsilon > 0$, there is a chiral phase $F = F(r)$ satisfying the above boundary conditions. The asymptotic forms of such a solution are given by equation (2.2) in Balakrishna *et al.* (1991). From these asymptotic solutions we immediately see that for $\varepsilon \to 0$ the chiral phase diverges at the lower limit.

2. CONSTANT-CUTOFF STABILIZATION

The chiral soliton with baryon number $n = 1$ is given by (1.4), where $F = F(r)$ is the radial chiral phase function satisfying the boundary conditions $F(0) = -\pi$ and $F(\infty) = 0$.

Substituting (1.4) into (1.1), we obtain the static energy of the chiral baryon

$$E_0 = \frac{\pi}{2} F_\pi^2 \int_{\varepsilon(t)}^{\infty} dr \left[r^2 \left(\frac{dF}{dr} \right)^2 + 2 \sin^2 F \right] \tag{2.1}$$

In (2.1) we avoid the singularity of the profile function $F = F(r)$ at the origin by introducing the cutoff $\varepsilon(t)$ at the lower boundary of the space interval $r \in [0, \infty]$, i.e., by working with the interval $r \in [\varepsilon, \infty]$. The cutoff itself is introduced following Balakrishna *et al.* (1991) as a dynamic time-dependent variable.

From (2.1) we obtain the following differential equation for the profile function $F = F(r)$:

$$\frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) = \sin 2F \tag{2.2}$$

with the boundary conditions $F(\varepsilon) = -\pi$ and $F(\infty) = 0$, such that the correct soliton number is obtained. The profile function $F = F[r; \varepsilon(t)]$ now depends implicitly on time t through $\varepsilon(t)$. Thus in the nonlinear σ-model Lagrangian

$$L = \frac{F_\pi^2}{16} \int \mathrm{Tr}(\partial_\mu U \, \partial^\mu U^+) \, d^3x \tag{2.3}$$

we use the ansätz

$$U(\mathbf{r}, t) = A(t) U_0(\mathbf{r}, t) A^+(t), \qquad U^+(\mathbf{r}, t) = A(t) U_0^+(\mathbf{r}, t) A^+(t) \tag{2.4}$$

where

$$U_0(\mathbf{r}, t) = \exp\{ i \boldsymbol{\tau} \cdot \mathbf{r}_0 F[r; \varepsilon(t)] \} \tag{2.5}$$

The static part of the Lagrangian (2.3), i.e.,

$$L = \frac{F_\pi^2}{16} \int \mathrm{Tr}(\nabla U \cdot \nabla U^+) \, d^3x = -E_0 \tag{2.6}$$

is equal to minus the energy E_0 given by (2.1). The kinetic part of the Lagrangian is obtained using (2.4) with (2.5) and it is equal to

$$L = \frac{F_\pi^2}{16} \int \mathrm{Tr}(\partial_0 U \, \partial_0 U^+) \, d^3x = bx^2 \, \mathrm{Tr}[\partial_0 A \, \partial_0 A^+] + c[\dot{x}(t)]^2 \tag{2.7}$$

where

$$b = \frac{2\pi}{3} F_\pi^2 \int_1^\infty \sin^2 F y^2\, dy, \qquad c = \frac{2\pi}{9} F_\pi^2 \int_1^\infty y^2 \left(\frac{dF}{dy}\right)^2 y^2\, dy \tag{2.8}$$

with $x(t) = [\varepsilon(t)]^{3/2}$ and $y = r/\varepsilon$. On the other hand, the static energy functional (2.1) can be rewritten as

$$E_0 = a x^{2/3}, \qquad a = \frac{\pi}{2} F_\pi^2 \int_1^\infty \left[y^2 \left(\frac{dF}{dy}\right)^2 + 2 \sin^2 F \right] dy \tag{2.9}$$

Thus the total Lagrangian of the rotating soliton is given by

$$L = c\dot{x}^2 - a x^{2/3} + 2 b x^2 \dot{\alpha}_\nu \dot{\alpha}^\nu \tag{2.10}$$

where $\mathrm{Tr}(\partial_0 A\, \partial_0 A^+) = 2\dot{\alpha}_\nu \dot{\alpha}^\nu$ and α_ν ($\nu = 0, 1, 2, 3$) are the collective coordinates defined as in Bhaduri (1988). In the limit of a time-independent cutoff ($\dot{x} \to 0$) we can write

$$H = \frac{\partial L}{\partial \dot{\alpha}^\nu} \dot{\alpha}^\nu - L = a x^{2/3} + 2 b x^2 \dot{\alpha}_\nu \dot{\alpha}^\nu = a x^{2/3} + \frac{1}{2 b x^2} J(J+1) \tag{2.11}$$

where $\langle \mathbf{J}^2 \rangle = J(J+1)$ is the eigenvalue of the square of the soliton laboratory angular momentum. A minimum of (2.11) with respect to the parameter x is reached at

$$x = \left[\frac{2}{3} \frac{ab}{J(J+1)}\right]^{-3/8} \Rightarrow \varepsilon^{-1} = \left[\frac{2}{3} \frac{ab}{J(J+1)}\right]^{1/4} \tag{2.12}$$

The energy obtained by substituting (2.12) into (2.11) is given by

$$E = \frac{4}{3} \left[\frac{3}{2} \frac{a^3}{b} J(J+1)\right]^{1/4} \tag{2.13}$$

This result is identical to the result obtained by Mignaco and Wulck, which is easily seen if we rescale the integrals a and b in such a way that $a \to (\pi/4) F_\pi^2 a$ and $b \to (\pi/4) F_\pi^2 b$ and introduce $f_\pi = 2^{-3/2} F_\pi$. However, in the present approach, as shown in Balakrishna *et al.* (1991), there is a profile function $F = F(y)$ with proper soliton boundary conditions $F(1) = -\pi$ and $F(\infty) = 0$ and the integrals a, b and c in (2.9)–(2.10) exist and are shown in Balakrishna *et al.* (1991) to be $a = 0.78\ \mathrm{GeV}^2$, $b = 0.91\ \mathrm{GeV}^2$, and $c = 1.46\ \mathrm{GeV}^2$ for $F_\pi = 186$ GeV.

Using (2.13), we obtain the same prediction for the mass ratio of the lowest states as Mignaco and Wulck (1989), which agrees rather well with the empirical mass ratio for the Δ resonance and the nucleon. Furthermore, using the calculated values for the integrals a and b, we obtain the nucleon mass $M(N) = 1167$ MeV, which is about 25% higher than the empirical value of 939 MeV. However, if we choose the pion decay constant equal to

$F_\pi = 150$ MeV, we obtain $a = 0.507$ GeV^2 and $b = 0.592$ GeV^2, giving exact agreement with the empirical nucleon mass.

Finally, it is of interest to know how large the constant cutoffs are for the above values of the pion decay constant in order to check if they are in the physically acceptable ballpark. Using (2.12), it is easily shown that for nucleons ($J = 1/2$) the cutoffs are equal to

$$\varepsilon = \begin{cases} 0.22 \text{ fm} & \text{for} \quad F_\pi = 186 \text{ MeV} \\ 0.27 \text{ fm} & \text{for} \quad F_\pi = 150 \text{ MeV} \end{cases} \tag{2.14}$$

From (2.14) we see that the cutoffs are too small to agree with the size of the nucleon (0.72 fm), as we should expect, since the cutoffs rather indicate the size of the quark-dominated bag in the center of the nucleon. Thus we find that the cutoffs are of reasonable physical size. Since the cutoff is proportional to F_π^{-1}, we see that the pion decay constant must be less than 57 MeV in order to obtain a cutoff which exceeds the size of the nucleon. Such values of the pion decay constant are not relevant to any physical phenomenon.

3. ELECTROMAGNETIC POLARIZABILITIES

The calculation of the static electromagnetic polarizabilities in the complete Skyrme model (1961, 1962) was first performed by Scherer and Mulders (1992). As argued in Scherer and Mulders (1992), the nucleon electromagnetic polarizabilities provide important information about the nucleon structure and it is of great interest to calculate these in the framework of the Skyrme model. The static electric (magnetic) polarizability of a baryon is a measure of how an electric (magnetic) dipole moment is induced by a constant external field: $\mathbf{p} = \alpha \mathbf{E}$ ($\boldsymbol{\mu} = \beta \mathbf{B}$). The dynamic or Compton polarizabilities α_d and β_d, obtained from a low-energy expansion of the Compton-scattering cross section, differ from the static polarizabilities due to relativistic and retardation effects related to the finite size of the nucleon. The dynamic electric polarizability of the neutron may also be obtained using Compton scattering on the neutron in the deuteron. The static electric polarizability for neutrons may be obtained from the scattering of low-energy neutrons on heavy nuclei. The most recent empirical values for electric and magnetic polarizabilities of nucleons, as found in Adkins *et al.* (1983), are given in Table I.

Theoretical investigations have used the nonrelativistic quark model (Drechsel and Russo, 1984; Schöberl and Leeb, 1986), chiral bag models (Hecking and Bertsch, 1981; Weiner and Weisse, 1985; Nyman, 1984), and, most recently, chiral perturbation theory (Bernard *et al.*, 1991, 1992). Their

Table I. Empirical Values for the Electric and Magnetic Polarizabilities of Nucleons in Units of 10^{-4} fm^3

Nucleon	α_d	β_d	α	β
p	10.9 ± 3.5	3.3 ± 3.5	–	–
n	$11.7^{+4.7}_{-11.7}$	–	12.0 ± 3.5	–

predictions vary significantly, and the range of variation can be found in Scherer and Mulders (1992).

In Dalarsson (1993) and here it is shown that the Skyrme model can be simplified by omitting the quartic Skyrme stabilizing term and performing the constant-cutoff quantum stabilization of the Skyrme soliton.

The chiral quantum model developed in Dalarsson (1993) gives the spectra of nonstrange and strange baryons in relatively good agreement with the empirical values. It is therefore of great interest to calculate the static electromagnetic polarizabilities of nucleons in the simplified Skyrme model as a further test of the self-consistency of the model. The aim of the present paper is to perform such a calculation and compare the results for the electromagnetic polarizabilities with the corresponding empirical values as well as the predictions of the complete Skyrme model found in Scherer and Mulders (1992).

4. THE ELECTROMAGNETIC INTERACTION IN THE SIMPLIFIED SKYRME MODEL

The Lagrangian density for the simplified $SU(2)$ Skyrme model has the form

$$\mathscr{L} = \frac{F_\pi^2}{16} \operatorname{Tr} \partial_\mu U \, \partial^\mu U^+ + \frac{m_\pi^2 F_\pi^2}{8} \operatorname{Tr}(U - 1) \tag{4.1}$$

where U is the usual $SU(2)$-skyrmion field, which in the static case has the form

$$U_0 = \exp[i\boldsymbol{\tau} \cdot \mathbf{r}_0 F(r)], \qquad \mathbf{r}_0 = \mathbf{r}/r \tag{4.2}$$

and $F(r)$ is a radial function which satisfies the nonlinear differential equation

$$\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = \sin(2F) + m_\pi^2 r^2 \sin F \tag{4.3}$$

with the boundary conditions $F(\varepsilon) = -\pi$ and $F(\infty) = 0$. Here F_π and m_π are the pion decay constant (empirical value $F_\pi = 186$ MeV) and pion mass (empirical value $m_\pi = 138$ MeV), respectively, while ε is the constant cutoff used in the constant-cutoff stabilization method (Dalarsson, 1993). In the $SU(2) \times SU(2)$ symmetric chiral model like the one described in the present paper, the Wess–Zumino action vanishes and does not contribute to the baryon dynamics.

In order to make the Lagrangian density (4.1) invariant under local electromagnetic gauge transformations, i.e.,

$$U \to \exp[-ieQ\chi(x)]\, U \exp[ieQ\chi(x)] \tag{4.4}$$

$$U^+ \to \exp[-ieQ\chi(x)]\, U^+ \exp[ieQ\chi(x)] \tag{4.5}$$

$$A_\mu \to A_\mu + \partial_\mu \chi(x) \tag{4.6}$$

where

$$Q = \begin{bmatrix} 2/3 & 0 \\ 0 & -1/3 \end{bmatrix} = \frac{1}{6} + \frac{1}{2}\tau_3, \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.7}$$

is the matrix of electric charge of u- and d-quarks, e is the elementary electric charge, and $\chi = \chi(x)$ is the local phase angle, we replace the ordinary derivatives in (4.1) by the covariant derivatives defined by

$$\partial_\mu U \to D_\mu U = \partial_\mu U + ieA_\mu[Q, U] \tag{4.8}$$

$$\partial_\mu U^+ \to D_\mu U^+ = \partial_\mu U^+ + ieA_\mu[Q, U^+] \tag{4.9}$$

where $[A, B] = AB - BA$. Expanding the gauge-invariant Lagrangian

$$\mathscr{L} = \frac{F_\pi^2}{16} \operatorname{Tr} D_\mu U D^\mu U^+ + \frac{m_\pi^2 F_\pi^2}{8} \operatorname{Tr}(U - 1) \tag{4.10}$$

up to terms of the second order in the elementary charge, i.e., up to e^2, we obtain

$$\begin{aligned} \mathscr{L} &= \mathscr{L}_0 + \mathscr{L}_{\text{Isovector}} + \mathscr{L}_{\text{Isoscalar}} \\ &= \frac{F_\pi^2}{16} \operatorname{Tr}(\partial_\mu U\, \partial^\mu U^+) + \frac{m_\pi^2 F_\pi^2}{8} \operatorname{Tr}(U - 1) \\ &\quad - eA_\mu J^\mu - \frac{e^2 F_\pi^2}{16} A_\mu A^\mu \operatorname{Tr}([Q, U][Q, U^+]) \\ &\quad - \frac{1}{2} eA_\mu B^\mu + i\frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma}\, \partial^\mu A^\nu A^\rho \operatorname{Tr}[(\partial^\sigma U U^+ + U^+ \partial^\sigma U)Q] \end{aligned} \tag{4.11}$$

with the Lagrangian of the free electromagnetic field omitted, and where

$$J_\mu = V_{3\mu} = -i\frac{F_\pi^2}{16}\,\mathrm{Tr}(\tau_3 U^+ \partial_\mu U + \tau_3 U \partial_\mu U^+) \tag{4.12}$$

is the isovector electromagnetic current density [see equation (3.5) in Dalarsson (1992) in the $SU(2) \times SU(2)$ case, with the Wess–Zumino term omitted and λ_3 replaced by τ_3], and

$$B_\mu = \frac{1}{24\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\; U^+ \partial^\nu U U^+ \partial^\alpha U U^+ \partial^\beta U \tag{4.13}$$

is the topological baryon current density. The fifth term in (4.11) proportional to the topological baryon current density (4.13) is the isoscalar electromagnetic interaction, linear in the electromagnetic field, and it is not gauge invariant. In order to make the total Lagrangian density gauge-invariant, we have added the last term in (4.11), which is quadratic in the electromagnetic fields.

5. STATIC ELECTRIC POLARIZABILITIES

The static electric polarizability is most easily extracted from the result for the shift in soliton energy in an external constant electric field $\mathscr{E} = \mathscr{E}\mathbf{z}_0$ with $A_\mu = (-z\mathscr{E}, 0, 0, 0)$ given by

$$\Delta E = -\frac{1}{2}\alpha\mathscr{E}^2 \tag{5.1}$$

The shift in soliton energy (5.1) is obtained by substituting the field $A_\mu = (-z\mathscr{E}, 0, 0, 0)$ into the interaction part of the Lagrangian density (4.11) and rotating the soliton field (4.2) as follows:

$$U(t) = A(t)U_0A^+(t) = \exp[i\tau^j R_{jk} r_0^k F(r)] \tag{5.2}$$

where R_{jk} is a 3×3 rotation matrix which actively rotates the isovector $\mathbf{r}_0 \sin F$ of the hedgehog soliton (5.2).

Thus we obtain the space-integrated Lagrangian to the second order in the elementary charge, i.e., up to e^2, in the form

$$L(t) = \frac{1}{2}\Omega\omega^2 - E_s + \frac{1}{2}g^e\mathscr{E}^2\left[1 - \frac{2}{5}D^{(2)}_{0,0}(\alpha, \beta, \gamma)\right] \tag{5.3}$$

where the inertia Ω and the static energy E_s of the soliton are given by

$$\Omega = \frac{2\pi}{3}F_\pi^2\int_\varepsilon^\infty dr\, r^2 \sin^2 F \tag{5.4}$$

$$E_s = \frac{\pi}{2} F_\pi^2 \int_\varepsilon^\infty dr \left[r^2 \left(\frac{dF}{dr} \right)^2 + 2 \sin^2 F + 4 m_\pi^2 r^2 \sin^2 \frac{F}{2} \right] \tag{5.5}$$

The Wigner rotation functions are of the form

$$D_{m,n}^{(j)}(\alpha, \beta, \gamma) = \langle j, m | \mathscr{R}(\alpha, \beta, \gamma) | j, n \rangle \tag{5.6}$$

with

$$\mathscr{R}(\alpha, \beta, \gamma) = \exp(-i\alpha J_x) \exp(-i\beta J_y) \exp(-i\gamma J_z)$$

The Euler angle α here should not be confused with the electric polarizability given in (5.1). Furthermore, the vector $\boldsymbol{\omega}$ is the angular velocity of the global rotation, defined by

$$\dot{R}_{jk} R_{mk} = -\varepsilon_{jmn} \omega_n \tag{5.7}$$

and in deriving equation (5.3) we used the relation

$$R_{33}^2 = \frac{1}{3} + \frac{2}{3} D_{0,0}^{(2)}(\alpha, \beta, \gamma) \tag{5.8}$$

From (5.3) we see that the terms linear in the electromagnetic fields in the Lagrangian density (4.11) do not contribute to the space-integrated Lagrangian (5.3), since the space integrals of the functions $A_0(\mathbf{x}) J^0(\mathbf{x}, t)$ and $A_0(\mathbf{x}) B^0(\mathbf{x}, t)$ are integrals of odd functions over symmetric intervals. Furthermore, the last term in (4.11) does not contribute to the static Lagrangian, since in the case of a constant electric field the indices ρ and ν in that term must both be zero, such that the antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ vanishes. Thus the only contributing term is the fourth term in (4.11), which gives rise to the following value of the parameter g^e in equation (5.3):

$$g^e = \frac{2\pi e^2}{9} F_\pi^2 \int_\varepsilon^\infty dr\, r^4 \sin^2 F \tag{5.9}$$

The momentum conjugate to the angular velocity $\boldsymbol{\omega}$ is the isospin $\mathbf{T}$, given by

$$\mathbf{T} = \frac{\delta L}{\delta \boldsymbol{\omega}} = \Omega \boldsymbol{\omega} \tag{5.10}$$

The Hamiltonian corresponding to the Lagrangian (5.3) is given by

$$\begin{aligned} H &= \mathbf{T} \cdot \boldsymbol{\omega} - L = H_0 + H_I \\ &= \frac{1}{2\Omega} \mathbf{T}^2 + E_s - \frac{1}{2} g^e \mathscr{E}^2 \left[1 - \frac{2}{5} D_{0,0}^{(2)}(\alpha, \beta, \gamma) \right] \end{aligned} \tag{5.11}$$

The first two terms of (5.11) constitute the Hamiltonian of the rotating soliton with the electric field switched off. The third term may be treated as the perturbation due to the presence of the electric field. The unperturbed Hamiltonian (H_0) is quantized by means of the collective coordinates, and the eigenfunctions of H_0 as the functions of the Euler angles are given by

$$\langle \alpha, \beta, \gamma | T = J, T_3, J_3 \rangle = (-1)^{J+J_3} \left[\frac{2J+1}{8\pi^2} \right]^{1/2} D^{(J)}_{T_3, -J_3}(\alpha, \beta, \gamma) \tag{5.12}$$

The perturbation Hamiltonian H_I commutes with T_3 and J_3 and it is therefore already diagonalized, such that the energy shift due to the presence of the electric field is easily calculated to read

$$\Delta E = \langle T = J, T_3, J_3 | H_I | T = J, T_3, J_3 \rangle = -\frac{1}{2} g^e \mathscr{E}^2 \tag{5.13}$$

and the polarizability $\alpha = g^e$ is equal to the parameter g^e given by (5.9).

From (5.13) we see that the electric polarizability is the same for all nucleon states regardless of their spin and isospin quantum numbers. This property is not shared by the Δ-particles, where the polarizability depends on their spin and isospin quantum numbers. Thus, using the result

$$\begin{aligned} &\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta \, d\beta \int_0^{2\pi} d\gamma \, \langle \alpha, \beta, \gamma | T = J, T_3, J_3 \rangle \\ &\quad \times D^{(2)}_{0,0} \langle \alpha, \beta, \gamma | T = J, T_3, J_3 \rangle \\ &= \frac{[3T_3^2 - T(T+1)][3J_3^2 - T(T+1)]}{(2T-1)T(T+1)(2T+3)} \end{aligned} \tag{5.14}$$

we obtain for Δ particles

$$\alpha = \begin{cases} g^e\left(1 - \dfrac{2}{25}\right), & |T_3| = |J_3| \\ g^e\left(1 + \dfrac{2}{25}\right), & |T_3| \neq |J_3| \end{cases} \tag{5.15}$$

In order to perform the numerical calculations of the electric polarizabilities of nucleons and Δ particles we must calculate the parameter g^e given by (3.8). Since $m_\pi^2 \varepsilon^2 \approx 0.02 \ll 1$, the symmetry-breaking term, proportional to m_π^2, constitutes only a few percent of the total energy and the chiral $SU(2) \times SU(2)$ symmetry is approximately well satisfied. However, in the present paper it is not possible to use the limit $m_\pi \to 0$, since for $m_\pi \to 0$ the electromagnetic polarizabilities diverge as $1/m_\pi$, as argued in Scherer and Mulders (1992). Therefore we use a profile $F = F(r)$, obtained for $m_\pi \neq 0$, to calculate the parameter g^e from equation (5.9),

$$g^e = \frac{2\pi e^2}{9} F_\pi^2 \varepsilon^5 \int_1^\infty dy\, y^4 \sin^2 F \approx 22.6 \times 10^{-4}\ \text{fm}^3 \tag{5.16}$$

where, following Scherer and Mulders (1992), we used Gaussian units ($e^2 = 1/137$). Using the result (5.16), we can now calculate the numerical values of the static electric polarizabilities of nucleons and Δ particles.

6. STATIC MAGNETIC POLARIZABILITIES

The static magnetic polarizability is extracted from the result for the shift in soliton energy in an external constant magnetic field $\mathbf{B} = B\mathbf{z}_0$ with $A_\mu = (0, 1/2\,\mathbf{B} \times \mathbf{r})$ given by

$$\Delta E = -\frac{1}{2}\beta B^2 \tag{6.1}$$

The shift in soliton energy (6.1) is obtained by substituting the field $A_\mu = (0, \frac{1}{2}\mathbf{B} \times \mathbf{r})$ into the interaction part of the Lagrangian density (4.11) and rotating the soliton field (4.2) according to equation (5.2). Thus we obtain the space-integrated Lagrangian to the second order in the elementary charge, i.e., up to e^2, in the form

$$L(t) = \frac{1}{2}\Omega\omega^2 - E_s - \frac{1}{2}eB\left[\Omega D_{0,0}^{(1)}(\alpha, \beta, \gamma) + \frac{1}{3}\langle r^2\rangle_B \omega^j R_{j3}\right] - \frac{1}{4}g^e B^2\left[1 + \frac{2}{5}D_{0,0}^{(2)}(\alpha, \beta, \gamma)\right] \tag{6.2}$$

where the inertia Ω and the static energy E_s of the soliton are given by (5.4) and (5.5), respectively, and where the parameter g^e is given by (5.9) or (5.16). The mean square radius $\langle r^2\rangle_B$ is given by

$$\langle r^2\rangle_B = -\frac{2}{\pi}\int_\varepsilon^\infty dr\, r^2 \sin^2 F \frac{dF}{dr} \tag{6.3}$$

In deriving the result (6.2) we used the result (5.8), the result

$$R_{33} = D_{0,0}^{(1)}(\alpha, \beta, \gamma) \tag{6.4}$$

and the fact that the last term in the Lagrangian density (4.11) does not contribute to the static magnetic polarizability either, as argued in Atkins *et al.* (1983). From (6.2) we see that the Lagrangian function contains both linear and quadratic terms in the magnetic field.

The isospin is now given by

$$T_j = \frac{\delta L}{\delta \omega^j} = \Omega \omega_j - \frac{1}{6} eB \langle r^2 \rangle_B R_{j3} \tag{6.5}$$

The Hamiltonian corresponding to the Lagrangian (6.2) is given by

$$H = T_j \omega^j - L = H_0 + H_{I1} + H_{I2} \tag{6.6}$$

where

$$H_O = \frac{1}{2\Omega} \mathbf{T}^2 + E_s \tag{6.7}$$

$$H_{I1} = \frac{1}{2} eB\Omega D^{(1)}_{0,0}(\alpha, \beta, \gamma) + \frac{eB}{6\Omega} \langle r^2 \rangle_B T^j R_{j3} \tag{6.8}$$

$$H_{I2} = \frac{1}{2} g^m B^2 \left[1 + \frac{2}{5} D^{(2)}_{0,0}(\alpha, \beta, \gamma) \right] + \frac{e^2 B^2}{72\Omega} \langle r^2 \rangle_B^2 R_{j3} R_{j3} \tag{6.9}$$

where we have introduced a magnetic parameter g^m as follows:

$$g^m = \frac{1}{2} g^e = 11.3 \times 10^{-4}\ \mathrm{fm}^3 \tag{6.10}$$

The two terms in H_{I1} are interaction terms of the magnetic field with the isovector and isoscalar magnetic moment. At this point there is a significant difference between the present model and the complete Skyrme model, since here the nonadiabatic correction to the isovector magnetic moment, which was neglected in Scherer and Mulders (1992) using the N_C^{-1} expansion in the large-N_C limit, does not appear at all.

The isospin generally does not commute with the rotation matrices R_{j3} occurring in the interaction terms (6.8) and (6.9). Therefore we have an ordering ambiguity in the second term of H_{I1}, as argued in Scherer and Mulders (1992). However, using the N_C^{-1} expansion in the large-N_C limit, we see that the first terms of both H_{I1} and H_{I2} are of the order N_C, while the second terms of both H_{I1} and H_{I2} are of the order N_C^{-1}. Thus in the large-N_C limit we obtain

$$H_{I1} = \frac{1}{2} eB\Omega D^{(1)}_{0,0}(\alpha, \beta, \gamma) \tag{6.11}$$

$$H_{12} = \frac{1}{2} g^m B^2 \left[1 + \frac{2}{5} D^{(2)}_{0,0}(\alpha, \beta, \gamma) \right] \tag{6.12}$$

with g^m given by equation (6.10). The energy shift (6.1) in the second-order perturbation theory is

$$\Delta E_\lambda = \langle\lambda|H_{I2}|\lambda\rangle + \sum_{\lambda \neq \lambda'} \frac{|\langle\lambda|H_{I1}|\lambda'\rangle|^2}{E_\lambda - E_{\lambda'}} \tag{6.13}$$

where $\lambda = \{J = T, J_3, T_3\}$.

Substituting (6.11) and (6.12) into (6.13), we obtain the energy shift for nucleons

$$\Delta E = \frac{1}{2}\left[-g^m + \frac{e^2\Omega^2}{9(M_\Delta - M_N)}\right]B^2 \tag{6.14}$$

where we used the argument that the states with $J = T > 3/2$ are spurious states for $N_C = 3$, such that the sum in (6.13) includes the contributions of the Δ particles and vice versa. Thus the magnetic polarizability of the nucleon is given by

$$\beta = \beta_{\text{dia}} + \beta_{\text{para}} = -g^m + \frac{e^2\Omega^2}{9(M_\Delta - M_N)} \tag{6.15}$$

In deriving the paramagnetic magnetic polarizability, we used the expression

$$\left|\left\langle \frac{3}{2}, J_3, T_3 \middle| D^{(1)}_{0,0}(\alpha, \beta, \gamma) \middle| \frac{3}{2}, J_3, T_3 \right\rangle\right|^2 = \frac{2}{9} \tag{6.16}$$

Similarly to the static electric case, all nucleon states have the same magnetic polarizability, while the magnetic polarizabilities of Δ particles depend on the spin and isospin quantum numbers. Thus we have

$$\beta = \begin{cases} -g^m\left(1 + \frac{2}{25}\right), & |T_3| = |J_3| = \frac{3}{2} \\ -g^m\left(1 + \frac{2}{25}\right) - \frac{e^2\Omega^2}{9(M_\Delta - M_N)}, & |T_3| = |J_3| = \frac{1}{2} \\ -g^m\left(1 - \frac{2}{25}\right), & |T_3| \neq |J_3| \end{cases} \tag{6.17}$$

If the states with $J = T = 5/2$ are taken into account, each delta polarizability picks up an additional paramagnetic term similar to the Δ-particle contribution to the magnetic polarizability of the nucleon states (6.15). In order to calculate the static magnetic polarizabilities of nucleons and Δ particles we must calculate the second term of (6.15), i.e.,

$$\beta_{\text{para}} = \frac{e^2\Omega^2}{9(M_\Delta - M_N)} = 3.4 \times 10^{-4}\ \text{fm}^3 \tag{6.18}$$

Using the results (6.10) and (6.18), we can now calculate the numerical values of the static electric polarizabilities of nucleons and Δ particles.

Finally, in the present model the relation $\alpha = -2\beta_{\text{dia}}$ follows exactly from (6.10), in agreement with soliton models including vector mesons instead of the Skyrme quartic term, but in disagreement with the complete Skyrme model (Scherer and Mulders, 1992).

7. NUMERICAL RESULTS FOR STATIC ELECTROMAGNETIC POLARIZABILITIES

In the present section we discuss the numerical results for the static electromagnetic polarizabilities, in the simplified Skyrme model developed in the previous sections. The results for the electric polarizabilities are given in Table II.

From Tables I and II, we see that the present static electric polarizabilities are closer to the empirical values than those obtained in the complete Skyrme model (Adkins *et al.*, 1983), but the difference between the present model and the complete Skyrme model (Adkins, 1983) is rather small. Present static electric polarizabilities are still larger than the empirical values by a factor 2–3.

The results for magnetic polarizabilities are given in Table III. The diamagnetic term β_{dia} in equation (6.15) and the similar terms in equation (6.17) have the same behavior as the electric polarizabilities. It should,

Table II. Static Electric Polarizabilities of the Nucleon and the Δ Particles in Units of $10^{-4}\,\text{fm}^3$

Particle	α ($F_\pi = 186$ MeV)	α ($F_\pi = 186$ MeV)[a]	α ($F_\pi = 108$ MeV)[a]
Nucleon	22.6	33.7	26.4
$\Delta_{\|T_3\|=\|J_3\|}$	20.8	31.0	24.3
$\Delta_{\|T_3\|\neq\|J_3\|}$	24.4	36.4	28.5

[a]Adkins *et al.* (1983).

Table III. Static Magnetic Polarizabilities of the Nucleon and the Δ Particles in Units of $10^{4}\,\text{fm}^3$

Particle	β ($F_\pi = 186$ MeV)	β ($F_\pi = 186$ MeV)[a]	β ($F_\pi = 108$ MeV)[a]
Nucleon	−7.9	7.1	−7.1
$\Delta_{\|T_3\|=\|J_3\|=3/2}$	−12.2	−14.9	−11.6
$\Delta_{\|T_3\|=\|J_3\|=1/2}$	−15.6	−38.2	−17.9
$\Delta_{\|T_3\|\neq\|J_3\|}$	−10.4	−17.5	−13.6

[a]Adkins *et al.* (1983).

however, be noted that in the present model there is still the impossibility to simultaneously reproduce the empirical value for the isovector magnetic moment $1/3e\Omega$ and the Δ-nucleon mass difference $M_\Delta - M_N$, which are both found in the expression for the paramagnetic term β_{para}. The predictions given in Table III should therefore be treated as order-of-magnitude estimates, as argued in Scherer and Mulders (1992).

From Tables I and III we see that the present static magnetic polarizabilities are close to those obtained from the complete Skyrme model (Scherer and Mulders, 1992) for $F_\pi = 108$ MeV, while there are more significant differences compared to the results obtained in Scherer and Mulders (1992) for $F_\pi = 186$ MeV. Here the result obtained in Scherer and Mulders (1992) for $F_\pi = 108$ MeV is somewhat closer to the empirical value of the nucleon magnetic polarizability, but as an order-of-magnitude estimate the present result is relatively accurate.

Finally it should be noted that the results of both Tables II and III can be relatively easily reproduced from the corresponding tables in Scherer and Mulders (1992) when only the contributions of the second-order terms (using $F_\pi = 186$ MeV) are kept and α, β_{dia}, and β_{para} are calculated in the framework of the simplified Skyrme model presented in Dalarsson (1993) or in the present paper. Furthermore, many of the analytic results throughout the paper are more or less identical to those of Scherer and Mulders (1992). Nevertheless we have included most of these results in order to demonstrate the analytical simplicity of the present model as well as some of the important analytic improvements such as the ones discussed below equations (6.10) and (6.18).

8. CONCLUSIONS

The present paper shows the possibility of using the Skyrme model for calculation of the electromagnetic polarizabilities of nucleons and Δ particles, without use of the Skyrme stabilizing term, proportional to e^{-2}, which makes practical calculations more complicated and generates nonadiabatic corrections to the first-order isovector terms.

For such a simple model with only one arbitrary dimensional constant F_π, which is chosen equal to its empirical value $F_\pi = 186$ MeV, the accuracy in the prediction of the electromagnetic polarizabilities of nucleons and Δ particles is rather satisfactory. The results are at least as accurate as those of the complete Skyrme model, reported in Scherer and Mulders (1992), although they are larger by a factor 2–3 than the available empirical values.

Furthermore, the relation $\alpha = -2\beta_{\text{dia}}$ is satisfied without the complications discussed in Scherer and Mulders (1992).

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